Classical hardness of Learning with Errors

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Joint work with
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Our main results

Not quantum

GapSVP in dimension $\sqrt{n}$

A classical reduction from a worst-case lattice problem to the Learning with Errors problem with small modulus.

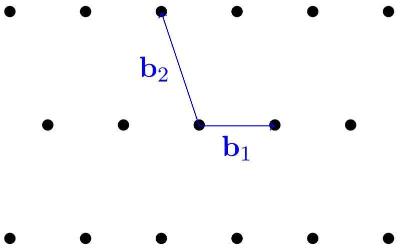
Dimension n

Polynomial in n

Outline

1. Lattices: definitions and problems
2. Lattice-based cryptography:
   LWE and a public-key encryption
3. Our main result:
   classical hardness of LWE for polynomial modulus

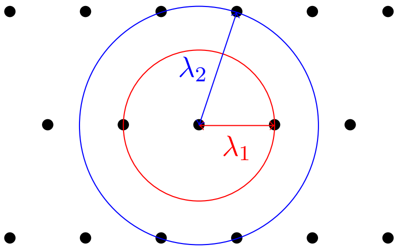
Lattices and problems



Lattice

$\mathcal{L}(\mathbf{B}) = \{\sum_{1=i}^{n} a_i \mathbf{b}_i, a_i \in \mathbb{Z}\}$, where the $(\mathbf{b}_i)_{1 \leq i \leq n}$'s, linearly independent vectors, are a basis of $\mathcal{L}(\mathbf{B})$.

Lattices and problems



Definitions:

- 1st minimum;
- 2nd minimum.

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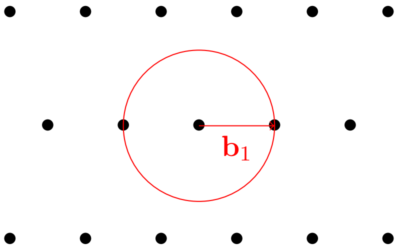

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- Shortest Vector Pbm. (computational or decisional version)

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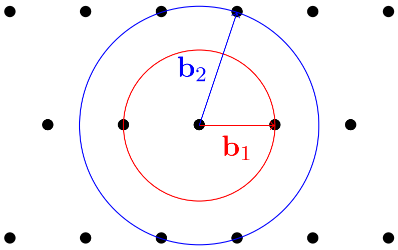

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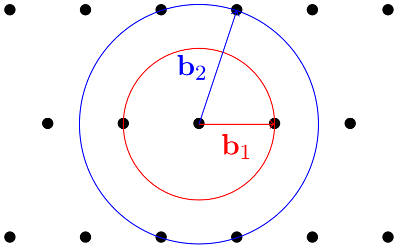
Lattices and problems



Definitions:

- 1st minimum;
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Problems :

- Shortest Vector Pbm. (computational or decisional version)
- Shortest Independent Vectors Pbm.
- Approximation factor: γ.

Conjecture

There is no polynomial time algorithm that approximates these lattice problems to within polynomial factors.

LWE-based cryptography

From basic to very advanced primitives

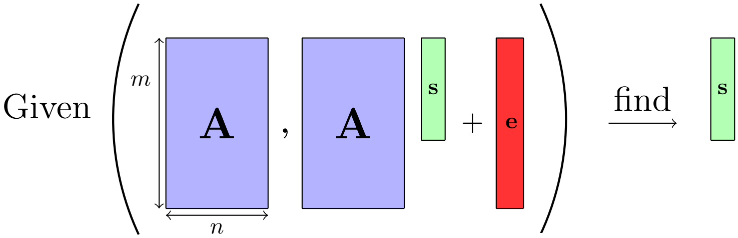
- Public key encryption
  [Regev 2005, ...];
- Identity-based encryption
  [Gentry, Peikert and Vaikuntanathan 2008, ...];
- Attribute-based encryption
  [Boyen 2013; Gorbunov, Vaikuntanathan and Wee 2013];
- Fully homomorphic encryption
  [Brakerski and Vaikuntanathan 2011, ...].

Advantages of LWE-based primitives

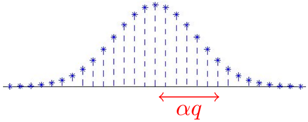
- Efficient, especially when the **modulus is polynomial**;
- Security proofs **from the hardness of LWE**;
- Likely to resist attacks from quantum computers.

The Learning With Errors problem [Regev05]

LWE_q^n (with m arbitrarily large)



- $\mathbf{A} \leftarrow U(\mathbb{Z}_q^{m\times n})$,
- $\mathbf{s} \leftarrow U(\mathbb{Z}_q^{n})$,
- $e \sim D_{\mathbb{Z}^m,\alpha q}$ with $\alpha = o(1)$.



Discrete Gaussian error

Decision version: Distinguish from $(\mathbf{A}, \mathbf{b})$ with $\mathbf{b}$ uniform.

LWE: solve a linear system with noise

Find $(s_1, s_2, s_3, s_4, s_5)$ such that:

$$\begin{aligned}
s_1 + 22s_2 + 17s_3 + 2s_4 + s_5 &\approx 16 \mod 23 \\
3s_1 + 2s_2 + 11s_3 + 7s_4 + 8s_5 &\approx 17 \mod 23 \\
15s_1 + 13s_2 + 10s_3 + s_4 + 22s_5 &\approx 3 \mod 23 \\
17s_1 + 11s_2 + s_3 + 10s_4 + 3s_5 &\approx 8 \mod 23 \\
2s_1 + s_2 + 13s_3 + 6s_4 + 2s_5 &\approx 9 \mod 23 \\
4s_1 + 4s_2 + s_3 + 5s_4 + s_5 &\approx 18 \mod 23 \\
11s_1 + 12s_2 + 5s_3 + s_4 + 9s_5 &\approx 7 \mod 23
\end{aligned}$$

$\rightsquigarrow$ Arbitrary number of equations.

Other interpretation: decoding a uniform linear code for the Euclidean distance.

An example of Public-Key Encryption[Regev 2005]

- **Parameters**: $n, m, q \in \mathbb{Z}$, $\alpha \in \mathbb{R}$,
- **Keys**: $\mathrm{sk} = \mathbf{s}$ and $\mathrm{pk} = (\mathbf{A}, \mathbf{b})$, with $\mathbf{b} = \mathbf{A}\mathbf{s} + \mathbf{e} \bmod q$ where $\mathbf{s} \hookleftarrow U(\mathbb{Z}_q^n)$, $\mathbf{A} \hookleftarrow U(\mathbb{Z}_q^{m \times n})$, $\mathbf{e} \hookleftarrow D_{\mathbb{Z}^m, \alpha q}$.
- **Encryption** $(M \in \{0,1\})$**:** Let $\mathbf{r} \hookleftarrow U(\{0,1\}^m)$,

$$\mathbf{u}^T = \mathbf{r} \, \mathbf{A}, \quad v = \mathbf{r} \, \mathbf{b} + \lfloor q/2 \rceil \cdot M$$

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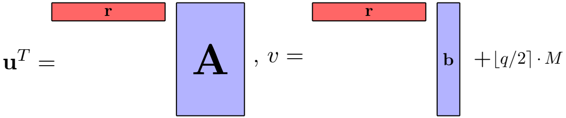
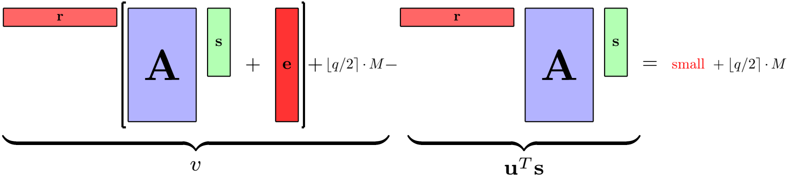


- **Decryption** of $(\mathbf{u}, v)$: compute $v - \mathbf{u}^T\mathbf{s}$,



If close from 0: return 0, if close from $\lfloor q/2 \rceil$: return 1.

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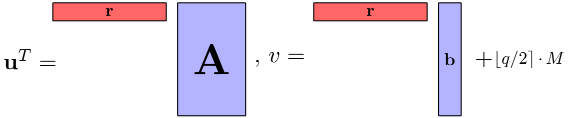

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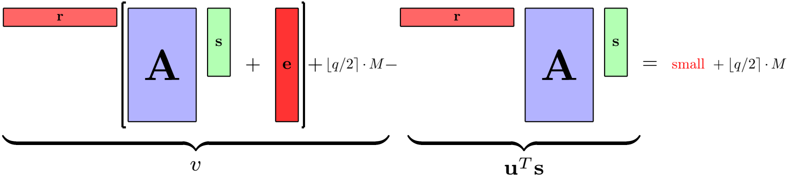

LWE hard $\Rightarrow$ Regev's scheme is "secure".

Prior reductions from worst-case lattice problem to LWE

- **[Regev05]**
  - A **quantum** reduction;
  - with q polynomial.

Quantum computer?

- **[Peikert09]**
  - A **classical** reduction;
  - with q exponential,

Inefficient primitives

- **[Peikert09]**
  - A **classical** reduction;
  - based on a non-standard lattice problem;
  - with q polynomial.

Hardness?

Prior reductions from worst-case lattice problem to LWE

- **[Regev05]**
  - A **quantum** reduction;
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- **[Peikert09]**
  - A **classical** reduction;
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- **[Peikert09]**
  - A **classical** reduction;
  - based on a non-standard lattice problem;
  - with q polynomial.

Our main result

- A **classical** reduction,
- from a standard worst-case lattice problem,
- with q polynomial.

Main component in the proof: a self reduction

- Recall that [Peikert09] already showed hardness of LWE with q exponential.

How do we obtain a hardness proof for q polynomial?

Main component in the proof: a self reduction

- Recall that [Peikert09] already showed hardness of LWE with q exponential.

How do we obtain a hardness proof for q polynomial?

- All we have to do is show the following reduction:

  A reduction from LWE with modulus q exponential to LWE with modulus p polynomial.

Modulus Switching

A reduction from LWE with modulus q to LWE with modulus p.

How to map $(\mathbf{A}, \mathbf{A}\mathbf{s} + \mathbf{e}) \bmod q$ to $(\mathbf{A}', \mathbf{A}'\mathbf{s} + \mathbf{e}') \bmod p$?

- Transform $\mathbf{A} \hookleftarrow U(\mathbb{Z}_q^{m \times n})$ to $\mathbf{A}' \hookleftarrow U(\mathbb{Z}_p^{m \times n})$;
  First idea: $\mathbf{A}' = \lfloor \frac{p}{q} \mathbf{A} \rceil$?

Modulus Switching

A reduction from LWE with modulus q to LWE with modulus p.

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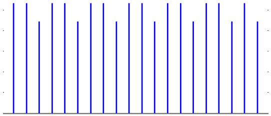
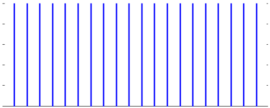
  First idea: $\mathbf{A}' = \lfloor \frac{p}{q} \mathbf{A} \rceil$?
- Two main problems:
  1. The distribution is not uniform:



A naive rounding introduces artefacts.

solution



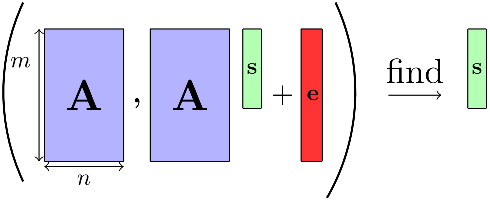
Add a **Gaussian rounding** to smooth the distribution: $\mathbf{A}' = \frac{p}{q}\mathbf{A} + \mathbf{R}$.

  2. In $\mathbf{A}'\mathbf{s} + \mathbf{e}'$, the rounding errors gets multiplied by the secret $\mathbf{s}$ (which is uniform is $\mathbb{Z}_q^n$).

From large to small secret

From LWE with arbitrary secret to LWE with binary secret.

- Inspired by ideas from cryptography (prior reduction by **[Goldwasser, Kalai, Peikert and Vaikuntanathan 2010]**) ; but different and stronger techniques.
- Definition of LWE:



- From $\mathbf{s}$ uniform in $\mathbb{Z}_q^n$ to $\mathbf{s}$ uniform in $\{0,1\}^n$.
- **Consequence:** this reduction expands the dimension from n to $n \log q$.

Summary of our new hardness proof of LWE

Our main result

A classical reduction from GapSVP in dimension $\sqrt{n}$ to LWE in dimension n with poly(n) modulus.

Reductions of the proof:

Problem	Dimension	Modulus	Secret	
GapSVP	$\sqrt{n}$			
$\downarrow_0$				**[Peikert09]**
LWE	$\sqrt{n}$	large	$\mathbb{Z}_q^{\sqrt{n}}$	
$\downarrow_1$				**New**
LWE	n	large	small	
$\downarrow_2$				**New**
LWE	n	poly(n)	in $\mathbb{Z}_q^n$	

Conclusion

Our main result

A classical reduction from GapSVP in dimension $\sqrt{n}$ to LWE in dimension n with $\mathrm{poly}(n)$ modulus.

Other results

The hardness of LWE_q^n is a function of $n \log q$.

Open problems:

Is there a classical reduction as good as the one in **[Regev05]**?

1. We lose a quadratic term in the dimension;
2. We do not have the same hard problem on lattices than Regev.